# Calibration Formula of Rheological Parameters of Bingham Fluid in Couette Rheometer

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## 1 Introduction

Bingham plastic (Bingham 1922) is a non-Newtonian fluid characterized with the yield stress. It has practical applications for the modeling in many fields, e.g., food engineering, polymer modeling, hydraulic engineering, and geological modeling. Different from Newtonian fluid, Bingham plastic behaves like a solid if the stress in fluid is less than yield stress, but likes a fluid if stress is greater than yield stress. To interpret the relationship between shear stress $\tau_{ij}$ and strain-rate tensor $e_{ij}$, Bingham plastic (Prager 1961) can be expressed as

$$\tau_{ij} = \left(\frac{\tau_0}{E} + 2\mu\right) e_{ij} \text{ , for } |\tau| \geq \tau_0 \text{, ............................ (1)}$$

$$E = 0 \text{ , for } |\tau| < \tau_0 \text{, ............................................. (2)}$$

where $(i, j) = (r, \theta)$ in our study; $\tau_0$ and $\mu$ are yield stress and dynamic viscosity respectively, and

$$E = \left(\frac{1}{2} e_{ij} e_{ij}\right)^{1/2} \text{ and } |\tau| = \left(\frac{1}{2} \tau_{ij}\tau_{ij}\right)^{1/2} \text{. ....................(3)}$$

Since rheological parameters ( $\tau_0$ , $\mu$ ) are essential in modeling, different types of rheometer have been developed for non-Newtonian fluid (Ferguson and Kemb³owski 1991). The Couette rheometer, which consists of two concentric cylinders, is probably one of the most commonly used types (Ferguson and Kemb³owski 1991; Dontula et al. 2005). This paper is to derive the calibration formula for the rheological parameters of Bingham fluid in Couette rheometer. We only consider steady Bingham fluid motion between two concentric cylinders with no thixotropic effect. Also, the cylinders are assumed to be infinitely long, so the end-effect can be neglected here. Thermal effect on rheological parameters shall also be neglected, so the yield stress $\tau_0$ and viscosity $\mu$ can be considered as constants.

With the assumption that two cylinders are placed vertically, the flow condition can be considered as two dimensional. Figure 1 shows the configurations of two cylinders and the coordinate system. The outer cylinder (with radius $R_2$ ) is stationary, but the inner one (with radius $R_1$ ) rotates counterclockwise at a constant rotating speed $\Omega$ . As can be shown later, stress distribution has its maximum at the inner cylinder and decreases monotonically towards outer cylinder. If the stress applied by the inner cylinder wall is less than yield stress, Bingham fluid would remain stationary due to the existence of yield stress. If the stress at the inner wall is greater than yield stress, fluid starts to move. As stress decreases away from the inner wall, it is possible that at some distance from the inner cylinder, stress becomes less than yield stress. The region further than this point also has stress less than yield stress, and fluid within this region does not move due to no-slip condition at the outer wall. This portion of stationary fluid is called the plug layer, and the flowing portion near the inner cylinder is called the shear layer. The shear layer thickness is defined as $\delta$ in Fig. 1. The flow condition including both shear and plug layer is defined as flow *with plug layer*. As rotating speed $\Omega$ increases, shear layer thickness increases. When $\Omega$ exceeds a critical rotational speed $\Omega_{cr}$ , all fluid within the gap is yielded and plug layer disappears. This flow condition is referred to as flow *without plug layer*. Both shear layer thickness $\delta$ and critical rotational speed $\Omega_{cr}$ are functions of rheological properties and inner cylinder rotating speed $\Omega$ . We shall derive solutions of the two different flow conditions separately. Besides, the effects of radius ratio and Bingham number are discussed. The rheometer with radii ratio very close to unity is discussed as a limiting case, and the result confirms with previous research. With steady flow motion solutions, we derive calibration formulas of rheological parameters for all values of radius ratio.

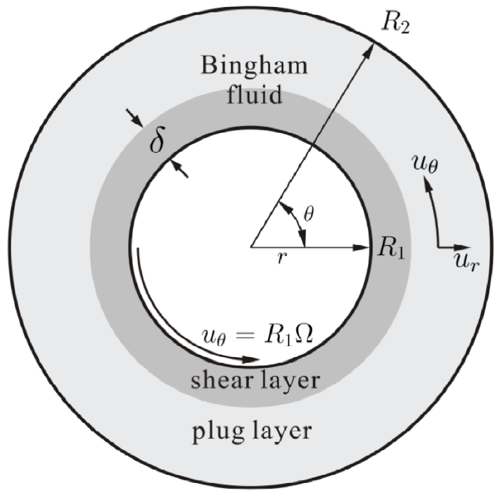


Fig. 1, Definition sketch of two cylinders and coordinate system. The clockwise direction of $\theta$ is defined as positive. The radii of the inner and outer cylinders are $R_1$ and $R_2$ respectively; $\delta$ is shear layer thickness, $u_r$ and $u_\theta$ are radial and angular velocity respectively. The inner cylinder rotates clockwise at a constant rotational speed $\Omega$ and the outer cylinder is stationary.

## 2 Calibration formulas

During experiments, one can vary the rotating speed ( $\Omega$ ) and then obtain the inner cylinder wall stress reading ( $\Gamma$ ). With many sets of these data and known cylinder geometry $(R_1, R_2)$, the rheological parameters $(\mu, \tau_0)$ can be calibrated. If the shear layer thickness $\delta$ can be measured either by dye tracking or video image processing, one can use the formulas in section 2.1 for calibration. Otherwise, one should refer to the equations in section 2.2. The calibration formulas in section 2.2 are derived using Least squares and perturbation methods.

### 2.1 Formula using shear layer thickness $\delta$

The yield stress:

$$\tau_0 = \frac{1}{N}\sum_{i=1}^{N}\frac{\Gamma_i}{(1+\delta_i/R_1)^2} \text{ ; ..........................................} \quad (4)$$

The dynamic viscosity:

$$\mu = \frac{1}{N}\sum_{i=1}^{N}\frac{\Gamma_i}{\Omega_i}\left\{\frac{1}{2} - \frac{1}{(1+\delta_i/R_1)^2}\left[\frac{1}{2} - \ln\left(1+\frac{\delta_i}{R_1}\right)\right]\right\} ,... \quad (5)$$

where $\Gamma_i$ =measured inner cylinder wall shear stress; $\delta_i$ =measured shear layer depth; $\Omega_i$ =given rotational speed of inner cylinder

### 2.2 Formula without shear layer thickness $\delta$

#### 2.2.1 Flow without plug layer

The yield stress:

$$\tau_0 = \left[\frac{R_2^2 - R_1^2}{2R_2^2 \ln\left(\frac{R_2}{R_1}\right)}\right]\left[\frac{\sum(\Gamma_i\Omega_i)\cdot\sum\Omega_i - \sum\Gamma_i\cdot\sum(\Omega_i^2)}{\left(\sum\Omega_i\right)^2 - N\sum(\Omega_i^2)}\right]$$

............ (6)

The dynamic viscosity:

$$\mu = \left(\frac{R_2^2 - R_1^2}{2R_2^2}\right)\left[\frac{N\sum(\Gamma_i\Omega_i) - \sum\Gamma_i\cdot\sum\Omega_i}{N\sum(\Omega_i^2) - \left(\sum\Omega_i\right)^2}\right] \text{..........} \quad (7)$$

where Σ(’") denotes the summation of all sets of measured data

#### 2.2.2 Flow with plug layer

The yield stress:

$$\tau_0 = \frac{\sum\Gamma_i\cdot\sum(\Omega_i^2) - \sum(\Gamma_i\Omega_i)\cdot\sum\Omega_i}{N\sum(\Omega_i^2) - \left(\sum\Omega_i\right)^2} - \varepsilon\left[\frac{\sum(\mathrm{M}_i\Omega_i)\cdot\sum\Omega_i - \sum\mathrm{M}_i\cdot\sum(\Omega_i^2)}{N\sum(\Omega_i^2) - \left(\sum\Omega_i\right)^2}\right] \text{.......} \quad (8)$$

The dynamic viscosity:

$$\mu = \frac{1}{2}\left[\frac{N\sum(\Gamma_i\Omega_i) - \sum\Gamma_i\cdot\sum\Omega_i}{N\sum(\Omega_i^2) - \left(\sum\Omega_i\right)^2}\right] + \frac{\varepsilon}{2}\left[\frac{\sum\mathrm{M}_i\cdot\sum\Omega_i - N\sum(\mathrm{M}_i\Omega_i)}{N\sum(\Omega_i^2) - \left(\sum\Omega_i\right)^2}\right] \text{.................} \quad (9)$$

where

$$\varepsilon = \frac{1}{N}\sum\left\{\frac{\tau_0^{(0)}\left[\ln\Gamma_i - \ln\left(\tau_0^{(0)}\right)\right]}{\Gamma_i}\right\} \text{ , and}$$

$$\mathrm{M}_i = \tau_0^{(0)}\ln\left(\frac{\Gamma_i}{\tau_0^{(0)}}\right).$$

## 3 Concluding remarks

We present the complete steady flow solutions of Bingham fluid without thixotropic effect contained between two concentric cylinders, which the inner cylinder rotates at a constant rotational speed but outer one stays stationary. The effects of radii ratio and Bingham number are discussed. We list two categories of calibration formulas based on the condition that if shear layer thickness is obtainable during experiments. Calibration formulas of rheological parameters are derived using Least squares and per@turbation methods for all values of radius ratio.


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